Logarithms and Exponentials

To define a logarithm it needs to be with respect to a base. Usually logs have base 10 or base $\underline{e}$ see below

This hard to grasp sentence defines a logarithm
"The logarithm of a number is the power to which the base must be raised to produce the number".

In powers of 10:

$100 = 10^2$, so if the base is 10, 2 is the power to which 10 must be raised to make 100.

So $\log_{10}(100) = 2$.

Similarly $\log_{10}(10{,}000) = 4$ as $10^4 = 10{,}000$.

For example in base 2 $\log_2(8) = 3$ as $2^3 = 8$
3 is the power to which 2 must be raised to make 8.

The exponent of 2 in $2^3 = 8$ is the number 3.
We can see how logs and exponents are related

$$\log_2 8 = 3 \iff 8 = 2^3$$

$$\log_{10} 10{,}000 = 4 \iff 10000 = 10^4$$

So in general $\boxed{\log_b x = y \iff x = b^y}$ Very important!

Log rules – these apply to any base (so base not written in)

$$\log a + \log b = \log ab \quad -(1)$$

$$\log a - \log b = \log(a/b) \quad -(2)$$

$$\log a^n = n \log a \quad -(3)$$

The magic number 'e'

When a graph is plotted of $y = a^x$ the gradient at any point turns out to be a constant multiple of a^x.

So $\frac{d(a^x)}{dx} = ka^x$. k depends on the value of a.

When $a = 2 \cdot 7182818\ldots\ldots$ it turns out that $k = 1$!
This value, which is a key number in Maths and Science is ~~given~~ given the symbol $\underline{\underline{e}}$.

$$\text{So } \frac{d}{dx}(e^x) = e^x, \text{ so } \frac{d^2}{dx^2}(e^x) = e^x \text{ etc.}$$

The importance of 'e' means that it is "natural" to define logarithms to base 'e'. They are called "natural" logarithms and ~~are~~ given the symbol $\ln$ – where the base is understood to be e.

Additional Information

$$\boxed{\begin{array}{l} 1. \ \dfrac{d(e^x)}{dx} = e^x \\ 2. \ \dfrac{d(e^{ax})}{dx} = ae^{ax} \end{array}}$$

3. You can take logarithms of both sides of an equation.

A common calculation:

Suppose $be^{ax} = c$

$$e^{ax} = \frac{c}{b}$$

Take $\ln$ of both sides

$$\ln(e^{ax}) = \ln c/b$$

$$ax = \ln c/b$$

By definition!

In general $\boxed{\log_b(b^y) = y.}$